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Dirichlet L-Functions and Character Power Sums

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A new representation for Dirichlet L-functions $L(s, \chi)$, valid for primitive characters χ modulo k and all complex s, is given in terms of the function $F(x, s)$ defined for real x and $R(s) > 1$ by the series $\sum_{n=1}^{\infty} e^{2n\pi ix}/n^s$. Evaluation of $L(s, \chi)$ for negative integer s leads to a class of identities relating mth power moments $\sum_{r=1}^{k-1} \chi(r) r^m$ with finite cotangent power sums. Special emphasis is given to the quadratic character $\chi(n) = (n \mid p)$, p an odd prime. A new proof of the functional equation for L-functions is also given.

1. INTRODUCTION

Let k be a positive integer and let χ be any character modulo k. The L-function $L(s, \chi)$, defined for $R(s) > 1$ by the Dirichlet series

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

can be expressed in terms of the Hurwitz zeta function $\zeta(s, a)$ by means of the formula

$$L(s, \chi) = k^{-s} \sum_{h=1}^{k-1} \chi(h)\, \zeta\left(s, \frac{h}{k}\right), \tag{1}$$

which is valid for all complex s [6, p. 113]. Here $\zeta(s, a)$ is the analytic continuation of the series

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \qquad R(s) > 1, \qquad 0 < a \leqslant 1.$$

In this paper we derive an analogous representation for $L(s, \chi)$, which we believe to be new, in terms of the function $\varphi(z, s)$ which is defined by the power-Dirichlet series

$$\varphi(z, s) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}. \tag{2}$$

This representation, stated below in Theorem 1, is valid only for primitive characters.

The series in (2) converges for all s if $|z| < 1$, for $R(s) > 0$ if $|z| = 1$, $z \neq 1$, and for $R(s) > 1$ if $z = 1$. The function $\varphi(z, s)$ has been studied by many writers, including Lambert, Legendre, Abel, Kummer, Appell, Lerch, Jonquière, Lindelöf, Wirtinger, Truesdell [7], and others. When $z = 1$, $\varphi(z, s)$ is the Riemann zeta function, $\zeta(s)$.

By use of the classic method of Riemann, $\varphi(z, s)$ can be extended to the whole s-plane by means of the contour integral [2, p. 28], [4, p. 144]

$$I(z, s) = \frac{1}{2\pi i} \int_C \frac{t^{s-1} z e^t}{1 - z e^t} \, dt,$$

where C is a loop in the t-plane which begins at $-\infty$, encircles the origin once in the positive direction, and returns to $-\infty$. If the z-plane is cut from 1 to ∞ along the positive real axis, then for every fixed z in the cut plane, $I(z, s)$ is an entire function of s, and we have

$$\varphi(z, s) = \Gamma(1 - s) \, I(z, s).$$

We are interested here in values of z on the unit circle $|z| = 1$, with $z \neq 1$. We let $z = e^{2\pi i x}$, where x is real, and we define $F(x, s) = \varphi(e^{2\pi i x}, s)$. Clearly, $F(x, s)$ is periodic in x with period 1. For $x \neq$ integer and $R(s) > 0$ we have the series representation

$$F(x, s) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n^s}. \tag{3}$$

When x is an integer we have $F(x, s) = \zeta(s)$.

In Section 2 we express $L(s, \chi)$ in terms of $F(x, s)$, and in Section 3 we use this representation to evaluate $L(s, \chi)$ when s is a negative integer or 0. This leads to a class of identities relating mth power moments

$$M_m(\chi) = \sum_{r=1}^{k-1} \chi(r) \, r^m \tag{4}$$

with cotangent power sums

$$T_m(\chi) = \sum_{r=1}^{k-1} \chi(r) \cot^m(\pi r/k) \tag{5}$$

for $m = 1, 2, 3, \ldots$.

In a study on quadratic residues, Lebesgue [5, p. 233] obtained the formula

$$\sum_{r=1}^{p-1} r(r \mid p) = -\tfrac{1}{2}\sqrt{p} \sum_{r=1}^{p-1} \cot(\pi r^2/p) \tag{6}$$

for primes $p \equiv 3 \pmod 4$, where $(r \mid p)$ is the quadratic character symbol of Legendre. In Section 4 we show that Lebesgue's formula is a very special case of the formulas in Section 3, obtained by choosing $m = 1$ and taking $\chi(n)$ to be the quadratic character $(n \mid p)$.

In Section 5 we again take $\chi(n) = (n \mid p)$, where p is an odd prime, and we obtain an alternate representation of $L(s, \chi)$ in terms of both $F(x, s)$ and $\zeta(s)$. This is used in Section 6 to evaluate $L(s, \chi)$ for negative integer values of s, giving us another set of identities relating the mth power moments

$$M_m = \sum_{r=1}^{p-1} r^m(r \mid p)$$

with the cotangent power sums

$$S_m = \sum_{r=1}^{p-1} \cot^m(\pi r^2/p).$$

For $p \equiv 3 \pmod 4$ these turn out to be the same identities obtained in Section 4, but for $p \equiv 1 \pmod 4$ they are different from those of Section 4.

Finally, in Section 7 we use the representation of Theorem 1 to give a new and short proof of the functional equation for L-functions.

2. A Representation Theorem for L-Functions

For any character χ modulo k let $G(m, \chi)$ denote the Gauss sum

$$G(m, \chi) = \sum_{h=1}^{k-1} \chi(h)\, e^{2\pi i m h/k},$$

and let $G(\chi) = G(1, \chi)$. Then we have:

THEOREM 1. *If χ is a primitive character modulo k, then for all complex s we have the representation*

$$G(\bar{\chi})\, L(s, \chi) = \sum_{h=1}^{k-1} \bar{\chi}(h)\, F\left(\frac{h}{k}, s\right). \tag{7}$$

Proof. We prove the result first for $R(s) > 1$ and then use analytic continuation to extend it for all s.

Starting with Eq. (3), we take $x = h/k$, multiply both members by $\bar{\chi}(h)$, and sum on h to obtain

$$\sum_{h=1}^{k-1} \bar{\chi}(h) F\left(\frac{h}{k}, s\right) = \sum_{n=1}^{\infty} n^{-s} \sum_{h=1}^{k-1} \bar{\chi}(h)\, e^{2\pi i n h/k} = \sum_{n=1}^{\infty} n^{-s} G(n, \bar{\chi}).$$

But since χ is a primitive character we have the factorization property [1, p. 312]

$$G(n, \bar{\chi}) = \chi(n)\, G(1, \bar{\chi}) = \chi(n)\, G(\bar{\chi}),$$

which holds for every n. Hence we obtain

$$\sum_{n=1}^{\infty} n^{-s} G(n, \bar{\chi}) = G(\bar{\chi}) \sum_{n=1}^{\infty} \chi(n)\, n^{-s} = G(\bar{\chi})\, L(s, \chi),$$

which proves (7).

3. Cotangent Identities for Character Power Sums

The two representations for $L(s, \chi)$ in (1) and (7) give us the identity

$$G(\bar{\chi})\, k^{-s} \sum_{r=1}^{k-1} \chi(r)\, \zeta\left(s, \frac{r}{k}\right) = \sum_{h=1}^{k-1} \bar{\chi}(h)\, F\left(\frac{h}{k}, s\right), \tag{8}$$

valid for all s. In this formula we substitute special values of s for which the functions $\zeta(s, a)$ and $F(x, s)$ can be evaluated explicitly. For example, if $s = 0$ we have [2, p. 27]

$$\zeta(0, a) = \tfrac{1}{2} - a.$$

From (2) we find $\varphi(z, 0) = z/(1 - z)$ if $|z| \leqslant 1, z \neq 1$. Taking $z = e^{2\pi i x}$, we obtain

$$F(x, 0) = -\frac{1}{2} + \frac{i}{2} \cot \pi x \tag{9}$$

if x is not an integer. Putting these values in (8) and using the relation $\sum_{r=1}^{k-1} \chi(r) = 0$ we obtain the identity

$$G(\bar{\chi}) \sum_{r=1}^{k-1} r\chi(r) = -\tfrac{1}{2} ik \sum_{h=1}^{k-1} \bar{\chi}(h) \cot(\pi h/k).$$

In the notation of (4) and (5) we have proved that

$$G(\bar{\chi})\,M_1(\chi) = -\tfrac{1}{2}ikT_1(\bar{\chi}). \tag{10}$$

The functions $\zeta(s, a)$ and $F(x, s)$ can also be evaluated when s is a negative integer. If $n \geqslant 1$ we have [2, p. 27]

$$\zeta(-n, a) = -B_{n+1}(a)/(n+1), \tag{11}$$

where $B_{n+1}(a)$ is a Bernoulli polynomial. The function values $F(x, -n)$ can be calculated recursively from the differential-difference equation

$$F(x, s-1) = \frac{1}{2\pi i}\frac{\partial F(x, s)}{\partial x}. \tag{12}$$

This follows from (3) for $R(s) > 2$, and by analytic continuation for all s. Using (12) repeatedly with the initial value $F(x, 0)$ in (9) we find

$$F(x, -n) = \frac{1}{(2\pi i)^n}\frac{d^n}{dx^n}\left(\frac{i}{2}\cot \pi x\right). \tag{13}$$

The first few values are

$$F(x, -1) = \frac{-1}{4}(1 + \cot^2 \pi x),$$

$$F(x, -2) = \frac{-i}{8}(2\cot \pi x + 2\cot^3 \pi x),$$

$$F(x, -3) = \frac{1}{16}(2 + 8\cot^2 \pi x + 6\cot^4 \pi x),$$

$$F(x, -4) = \frac{i}{32}(16\cot \pi x + 40\cot^3 \pi x + 24\cot^5 \pi x).$$

It is easy to prove by induction that for any $n \geqslant 1$, $F(x, -n)$ is a polynomial in $\cot \pi x$ of degree $n + 1$ given by

$$F(x, -n) = \frac{i^{n+1}}{2^{n+1}}\sum_{k=0}^{n+1} a_{n,k}\cot^k \pi x.$$

The coefficients $a_{n,k}$ are nonnegative integers which can be readily calculated from the following recursion formulas:

$$\begin{aligned}
a_{0,0} &= 0, \qquad a_{0,1} = 1,\\
a_{n,0} &= a_{n-1,1} \quad \text{for} \quad n \geqslant 1,\\
a_{n,k} &= (k-1)\,a_{n-1,k-1} + (k+1)\,a_{n-1,k+1} \quad \text{for} \quad 1 \leqslant k \leqslant n,\\
a_{n,n+1} &= n! \quad \text{for} \quad n \geqslant 1.
\end{aligned}$$

The coefficients for $n \leqslant 6$ are listed in Table I.

TABLE I

n	$a_{n,0}$	$a_{n,1}$	$a_{n,2}$	$a_{n,3}$	$a_{n,4}$	$a_{n,5}$	$a_{n,6}$	$a_{n,7}$
1	1	0	1					
2	0	2	0	2				
3	2	0	8	0	6			
4	0	16	0	40	0	24		
5	16	0	136	0	240	0	120	
6	0	272	0	1232	0	1680	0	720

Putting $s = -n$ in Theorem 1 and multiplying by $-(n+1)$ we find

$$k^n G(\bar{\chi}) \sum_{r=1}^{k-1} \chi(r)\, B_{n+1}\left(\frac{r}{k}\right) = -(n+1) \sum_{h=1}^{k-1} \bar{\chi}(h)\, F\left(\frac{h}{k}, -n\right), \quad (14)$$

for $n = 1, 2, 3, \ldots$. Taking $n = 1$ and using the formula

$$B_2(x) = x^2 - x + \tfrac{1}{6}$$

we are led to the relation

$$G(\bar{\chi})\, M_2(\chi) = kG(\bar{\chi})\, M_1(\chi) + \tfrac{1}{2}kT_2(\bar{\chi}).$$

Using (10) to express $G(\bar{\chi})\, M_1(\chi)$ in terms of $T_1(\bar{\chi})$ we find

$$G(\bar{\chi})\, M_2(\chi) = -\tfrac{1}{2}ik^2T_1(\bar{\chi}) + \tfrac{1}{2}kT_2(\bar{\chi}). \quad (15)$$

When $n = 2$ we have $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$. This leads to the formula

$$G(\bar{\chi})\, M_3(\chi) = \tfrac{3}{2}kG(\bar{\chi})\, M_2(\chi) - \tfrac{1}{2}k^2G(\bar{\chi})\, M_1(\chi) + \tfrac{3}{4}ik\{T_1(\bar{\chi}) + T_3(\bar{\chi})\}.$$

Using (10) and (15) we find

$$G(\bar{\chi})\, M_3(\chi) = i(\tfrac{3}{4}k - \tfrac{1}{2}k^3)\, T_1(\bar{\chi}) + \tfrac{3}{4}k^2T_2(\bar{\chi}) + \tfrac{3}{4}ikT_3(\bar{\chi}). \quad (16)$$

When $n = 3$ we have $B_4(x) = x^4 - 2x^3 + x^2 - 1/30$. When this is used in (14) we obtain

$$G(\bar{\chi})\, M_4(\chi) = 2kG(\bar{\chi})\, M_3(\chi) - k^2G(\bar{\chi})\, M_2(\chi) - 2kT_2(\bar{\chi}) - \tfrac{3}{2}kT_4(\bar{\chi}).$$

Using (15) and (16) we find

$$G(\bar{\chi})\, M_4(\chi) = \tfrac{1}{2}ik^2(3-k^2)\, T_1(\bar{\chi}) + k(k^2-2)\, T_2(\bar{\chi}) + \tfrac{3}{2}ik^2T_3(\bar{\chi}) - \tfrac{3}{2}kT_4(\bar{\chi}). \quad (17)$$

It is clear that, in general, the number $G(\bar{\chi})\, M_m(\chi)$ can be expressed as a linear combination of the cotangent power sums $T_1(\bar{\chi}),\dots, T_m(\bar{\chi})$, the coefficients being polynomials in k.

4. Quadratic Residue Power Sums

This section considers some special cases of the formulas in Section 3. We take $k = p$, where p is an odd prime, and let $\chi(n) = (n \mid p)$, the quadratic character modulo p, where it is understood that $(n \mid p) = 0$ if $n \equiv 0 \pmod{p}$. For this choice of χ the Gauss sum $G(\chi)$ is given by [1, p. 315]

$$G(\chi) = \sum_{r=1}^{p-1} (r \mid p)\, e^{2\pi i r/p} = \begin{cases} \sqrt{p} & \text{if } p \equiv 1 \pmod{4} \\ i\sqrt{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \tag{18}$$

Therefore the formulas in Section 3 split naturally into two cases, $p \equiv 1$ and $p \equiv 3 \pmod{4}$. We write M_m for $M_m(\chi)$ and T_m for $T_m(\chi)$ so that

$$M_m = \sum_{r=1}^{p-1} (r \mid p)\, r^m, \qquad T_m = \sum_{h=1}^{p-1} (h \mid p) \cot^m(\pi h/p).$$

Both sums M_m and T_m are real-valued. Using (18), and equating real and imaginary parts in Eq. (10), we obtain the pair of formulas

$$M_1 = T_1 = 0 \qquad \text{if} \qquad p \equiv 1 \pmod{4} \tag{19}$$

and

$$M_1 = -\tfrac{1}{2}\sqrt{p}\, T_1 \qquad \text{if } p \equiv 3 \pmod{4}. \tag{20}$$

It is easy to show that (20) contains Lebesgue's formula (6). Since the cotangent is an odd function with period π we have

$$\sum_{r=1}^{p-1} \cot(\pi r/p) = 0.$$

Therefore we have

$$\begin{aligned} T_1 &= \sum_{h=1}^{p-1} (h \mid p) \cot(\pi h/p) = \sum_{h=1}^{p-1} \{(h \mid p) + 1\} \cot(\pi h/p) \\ &= 2 \sum_{\substack{h=1 \\ (h \mid p)=1}}^{p-1} \cot(\pi h/p) = \sum_{r=1}^{p-1} \cot(\pi r^2/p), \end{aligned} \tag{21}$$

so Eq. (20) reduces to Lebesgue's formula (6).

The identities in Section 3 corresponding to $m = 2, 3, 4$ lead to the following pairs of formulas:

$$M_2 = \begin{cases} \frac{1}{2}\sqrt{p}\, T_2 & \text{if } p \equiv 1 \pmod 4 \\ -\frac{1}{2}p\sqrt{p}\, T_1 & \text{if } p \equiv 3 \pmod 4, \end{cases} \tag{22}$$

$$M_3 = \begin{cases} \frac{3}{4}p\sqrt{p}\, T_2 & \text{if } p \equiv 1 \pmod 4 \\ -\frac{1}{4}\sqrt{p}\,(2p^2 - 3)\, T_1 + \frac{3}{4}\sqrt{p}\, T_3 & \text{if } p \equiv 3 \pmod 4, \end{cases} \tag{23}$$

$$M_4 = \begin{cases} \sqrt{p}\,(p^2 - 2)\, T_2 - \frac{3}{2}\sqrt{p}\, T_4 & \text{if } p \equiv 1 \pmod 4 \\ -\frac{1}{2}\sqrt{p}\, p(p^2 - 3)\, T_1 + \frac{3}{2}p\sqrt{p}\, T_3 & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{24}$$

As a by-product we also obtain the relation $T_{2m+1} = 0$ for $p \equiv 1 \pmod 4$, and $T_{2m} = 0$ for $p \equiv 3 \pmod 4$.

5. Alternate Representation for $L(s, \chi)$ when $\chi(n) = (n \mid p)$.

Let p be an odd prime, let $\chi(n) = (n \mid p)$, let $G = G(1, \chi)$, and let $L(s) = L(s, \chi)$. Then we have:

THEOREM 2. *For all complex s we have the representation*

$$GL(s) = \sum_{h=1}^{p-1} F\left(\frac{h^2}{p}, s\right) + (1 - p^{1-s})\, \zeta(s). \tag{25}$$

Proof. Taking $k = p$ and $\chi(n) = (n \mid p)$ in Theorem 1 we have

$$GL(s) = \sum_{r=1}^{p-1} (r \mid p)\, F\left(\frac{r}{p}, s\right) = \sum_{\substack{r=1 \\ (r\mid p)=1}}^{p-1} F\left(\frac{r}{p}, s\right) - \sum_{\substack{r=1 \\ (r\mid p)=-1}}^{p-1} F\left(\frac{r}{p}, s\right). \tag{26}$$

Using (3) we have, for $R(s) > 1$,

$$\sum_{r=1}^{p} F\left(\frac{r}{p}, s\right) = \sum_{n=1}^{\infty} n^{-s} \sum_{r=1}^{p} e^{2\pi i n r/p} = p \sum_{\substack{n=1 \\ n \equiv 0 (\mathrm{mod}\, p)}}^{\infty} n^{-s} = p^{1-s}\zeta(s).$$

Therefore, by analytic continuation, for all s we have

$$p^{1-s}\zeta(s) = \sum_{r=1}^{p} F\left(\frac{r}{p}, s\right) = \sum_{\substack{r=1 \\ (r\mid p)=1}}^{p-1} F\left(\frac{r}{p}, s\right) + \sum_{\substack{r=1 \\ (r\mid p)=-1}}^{p-1} F\left(\frac{r}{p}, s\right) + F(1, s). \tag{27}$$

Adding (26) and (27) we obtain

$$GL(s) = 2 \sum_{\substack{r=1 \\ (r|p)=1}}^{p-1} F\left(\frac{r}{p}, s\right) + (1 - p^{1-s})\, \zeta(s)$$

$$= \sum_{h=1}^{p-1} F\left(\frac{h^2}{p}, s\right) + (1 - p^{1-s})\, \zeta(s),$$

which proves (25).

6. Further Cotangent Identities for Quadratic Characters

Using Eq. (1) we can rewrite the identity of Theorem 2 in the form

$$Gp^{-s} \sum_{r=1}^{p-1} (r \mid p)\, \zeta\left(s, \frac{r}{p}\right) = \sum_{h=1}^{p-1} F\left(\frac{h^2}{p}, s\right) + (1 - p^{1-s})\, \zeta(s).$$

In this formula we substitute $s = -n$ $(n = 0, 1, 2, 3, \ldots)$, using (11), and the relation [2, p. 34]

$$\zeta(-n) = -B_{n+1}/(n + 1),$$

where B_{n+1} is a Bernoulli number. This gives us

$$Gp^{n} \sum_{r=1}^{p-1} (r \mid p)\, B_{n+1}\left(\frac{r}{p}\right) = (p^{n+1} - 1)\, B_{n+1} - (n + 1) \sum_{h=1}^{p-1} F\left(\frac{h^2}{p}, -n\right). \tag{28}$$

Now we take $n = 0, 1, 2, 3$ and proceed as in Section 4 (omitting details) to obtain the following formulas:

$$GM_1 = -\tfrac{1}{2} ipS_1, \tag{29}$$

$$GM_2 = \tfrac{1}{2} pS_2 - \tfrac{1}{6} p(p - 1)(p - 2) - \tfrac{1}{2} ip^2 S_1, \tag{30}$$

$$GM_3 = -\tfrac{1}{4} p^2(p - 1)(p - 2) + i(\tfrac{3}{4} p - \tfrac{1}{2} p^3) S_1 + \tfrac{3}{4} p^2 S_2 + \tfrac{3}{4} ipS_3, \tag{31}$$

$$\begin{aligned} GM_4 = {} & \tfrac{1}{30} p(p - 1)(p - 2)(7 + 3p - 9p^2) + \tfrac{1}{2} ip^2(3 - p^2)\, S_1 \\ & + p(p^2 - 2)\, S_2 + \tfrac{3}{2} ip^2 S_3 - \tfrac{3}{2} pS_4 . \end{aligned} \tag{32}$$

Using the value of G in (18) we obtain the following pairs of formulas:

$$M_2 = \begin{cases} \frac{1}{2}\sqrt{p}\,S_2 - \frac{1}{6}\sqrt{p}\,(p-1)(p-2) & \text{if } p \equiv 1 \pmod 4 \\ -\frac{1}{2}p\sqrt{p}\,S_1 & \text{if } p \equiv 3 \pmod 4, \end{cases} \tag{33}$$

$$M_3 = \begin{cases} \frac{3}{4}\sqrt{p}\,pS_2 - \frac{1}{4}\sqrt{p}\,p(p-1)(p-2) & \text{if } p \equiv 1 \pmod 4 \\ \frac{3}{4}\sqrt{p}\,S_3 - \frac{1}{4}\sqrt{p}\,(2p^2-3)\,S_1 & \text{if } p \equiv 3 \pmod 4, \end{cases} \tag{34}$$

$$M_4 = \begin{cases} \frac{1}{30}\sqrt{p}\,(p-1)(p-2)(7+3p-9p^2) + \sqrt{p}\,(p^2-2)S_2 - \frac{3}{2}\sqrt{p}\,S_4 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } p \equiv 1 \pmod 4 \\ \frac{3}{2}\sqrt{p}\,pS_3 - \frac{1}{2}\sqrt{p}\,p(p^2-3)\,S_1 \qquad \text{if } p \equiv 3 \pmod 4. \end{cases}$$

We also obtain the relations

$$S_2 = \tfrac{1}{3}(p-1)(p-2) \qquad \text{if } p \equiv 3 \pmod 4$$

and

$$S_4 = \tfrac{1}{45}(p-1)(p-2)(p^2+3p-13) \qquad \text{if } p \equiv 3 \pmod 4.$$

By the same argument used in Eq. (21) to prove that $T_1 = S_1$ we see that the cotangent sums T_m and S_m are equal when m is odd. Therefore, for $p \equiv 3 \pmod 4$ the above formulas for $M_1, \ldots, M_4$ are the same as those obtained in Section 4. By comparing the formulas (33) through (35) with the corresponding results in Section 4 we also obtain the following cotangent identities, valid for primes $p \equiv 1 \pmod 4$:

$$T_2 = S_2 - \tfrac{1}{3}(p-1)(p-2) \qquad \text{if } p \equiv 1 \pmod 4$$

and

$$T_4 = S_4 - \tfrac{1}{45}(p-1)(p-2)(p^2+3p-13) \qquad \text{if } p \equiv 1 \pmod 4.$$

7. The Functional Equation for L-Functions

In this section we give a simple proof of the functional equation for L-functions [6, p. 307].

Theorem 3. *If χ is any primitive character modulo k, then for all complex s we have*

$$L(1-s,\chi) = (2\pi)^{-s}\Gamma(s)\,k^{s-1}\{e^{-is\pi/2} + \chi(-1)\,e^{is\pi/2}\}\,G(\chi)\,L(s,\bar{\chi}). \tag{36}$$

Proof. Our proof is based on the formula [2, p.31]

$$\zeta(1-s, x) = (2\pi)^{-s}\Gamma(s)\{e^{-is\pi/2}F(x, s) + e^{is\pi/2}F(-x, s)\}, \tag{37}$$

which is valid for $0 < x \leqslant 1$ and all complex s. This relation is usually attributed to Jonquière [4] who proved it in 1889. However, it is really an alternate form of a well-known formula of Hurwitz [3], proved in 1882, which states that [2, p. 26]

$$\zeta(s, x) = 2(2\pi)^{s-1}\,\Gamma(1-s)\sum_{n=1}^{\infty}\frac{\sin(2n\pi x + \pi s/2)}{n^{1-s}} \tag{38}$$

if $R(s) < 0$ and $0 < x \leqslant 1$. If we replace s by $1 - s$ and express $\sin(2n\pi x + \pi s/2)$ in terms of exponentials, Eq. (38) takes the form

$$\zeta(1-s, x) = (2\pi)^{-s}\,\Gamma(s)\left\{e^{-is\pi/2}\sum_{n=1}^{\infty}\frac{e^{2\pi inx}}{n^s} + e^{is\pi/2}\sum_{n=1}^{\infty}\frac{e^{-2\pi inx}}{n^s}\right\}$$

provided $R(s) > 1$. This clearly implies (37) for $R(s) > 1$, and the result holds for all s by analytic continuation.

To derive the functional equation (36) from (37) we simply take $x = h/k$ in (37), then multiply both members by $\chi(h)$ and sum on h. This gives us

$$\begin{aligned}
&\sum_{h=1}^{k-1}\chi(h)\,\zeta\left(1-s, \frac{h}{k}\right)\\
&\quad = \frac{\Gamma(s)}{(2\pi)^s}\left\{e^{-is\pi/2}\sum_{h=1}^{k-1}\chi(h)\,F\left(\frac{h}{k}, s\right) + e^{is\pi/2}\sum_{h=1}^{k-1}\chi(h)\,F\left(\frac{-h}{k}, s\right)\right\}\\
&\quad = \frac{\Gamma(s)}{(2\pi)^s}\{e^{-is\pi/2} + \chi(-1)\,e^{is\pi/2}\}\sum_{h=1}^{k-1}\chi(h)\,F\left(\frac{h}{k}, s\right),
\end{aligned} \tag{39}$$

since

$$\begin{aligned}
\sum_{h \bmod k}\chi(h)\,F\left(\frac{-h}{k}, s\right) &= \chi(-1)\sum_{h \bmod k}\chi(-h)\,F\left(\frac{-h}{k}, s\right)\\
&= \chi(-1)\sum_{h \bmod k}\chi(h)\,F\left(\frac{h}{k}, s\right).
\end{aligned}$$

Now we multiply both members of (39) by k^{s-1}, then use (1) on the left and Theorem 1 on the right to obtain (36).

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